Lecture 4: Infinite Series

$\{a_1, a_2, \dots, a_n\}$ finite sequence

$\{a_1, \dots a_n, \dots\}$ infinite sequence

$a_1 + a_2 + a_3 + \dots + a_n$ finite series

$a_1 + a_2 + a_3 + \dots + a_n + \dots$ infinite series

$\sum_{n=1}^{\infty} a_n$ $\qquad S_n = \sum_{i=1}^{n} a_i$

eg $\sum_{n=1}^{\infty} 0 = 0$ $\qquad \sum_{n=1}^{\infty} 1 = \infty$

If $\lim_{n\to\infty} S_n = L$ (finite)

then $\sum_{n=1}^{\infty} a_n$ convergent

S_n	$\lim_{n\to\infty}$	$\sum$
cosh	not exist	$>$ divergent
n	exist, ∞	$>$ divergent
L	finite	— convergent

eg $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)}$

$= (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots + (\frac{1}{n} - \frac{1}{n+1})$

$= 1 - \frac{1}{n+1}$

$\therefore \lim_{n\to\infty} (1 + \frac{1}{n+1}) = 1$ and we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and equals to 1.

$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_n$

eg $\sum_{n=1}^{\infty} a^n$, $a \neq 1$

$S_n = \frac{1-a^n}{1-a}$ (the sum of geometric sequence)

$\lim_{n\to\infty} \frac{1}{1-a}(1-a^n)$

If a^n tends to 0, then

$\lim_{n\to\infty} \frac{1}{1-a}(1-a^n) = \frac{1}{1-a}$, $|a|<1$

$\sum_{n=1}^{\infty} a^n = \frac{1}{1-a}$, convergent

$|a|>1$, $\lim_{n\to\infty} S_n$ does not exist ($a \leq -1$, $a > 1$)

$\sum_{n=1}^{\infty} a^n$, $|a|>1$ diverges

then, $\sum_{n=1}^{\infty} a^n = \begin{cases} \frac{1}{1-a} & |a|<1 \\ \text{divergent} & |a| \geq 1 \end{cases}$

Given that $\sum a_n$ and $\sum b_n$ are convergent series

then, $\sum (a_n + b_n) = \sum a_n + \sum b_n$

$\sum t a_n = t \sum a_n$, where t is a scalar.

If $\sum a_n$ converges $\Rightarrow a_n \to 0$, as $n \to \infty$

$\sum \frac{1}{n}$ divergent, but $\frac{1}{n} \to 0$

* Therefore, if the series is convergent, then $a_n \to 0$, as $n \to \infty$; however, $a_n \to 0$ when $n \to \infty$ says nothing.

- Positive series

$a_n \geq 0$, $\sum a_n = S_n$, we call that is positive series

- Comparison Test

eg $|S_n| \leq M \Rightarrow |S_n|$ converges $\quad |S_n| \geq S_n$

and $S_n \leq M$ because $|S_n| \leq M$

$\therefore S_n$ converges

eg positive $\sum a_n$, $a_n \leq b_n$ (exclusive finite number)

$\sum b_n$ converges

$\Rightarrow \sum a_n$ converges

$a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots + a_{2n} + a_{2n+1} + \cdots$

$\wedge\!| \quad \wedge\!| \quad \cdots \quad \wedge\!| \quad \wedge\!| \quad \cdots \quad \wedge\!| \quad \wedge\!| \quad \cdots \quad \wedge\!|$

Convergent (given) $b_1 + b_2 + \cdots + b_n + b_{n+1} + \cdots + b_{2n} + b_{2n+1} + \cdots$

* If infinity numbers $b_n \geq a_n$, and $\sum b_n$ converges, then $\sum a_n$ converges.

- Integral Test ($f'(x) < 0$)

If $f(x) \geq 0$ and decreasing for $x \geq 1$, $f(n) = a_n$

$\sum a_n$ converges $\Leftrightarrow \lim_{n\to\infty} \int_1^n f(x)\,dx = L$ $\quad \left(\lim_{n\to\infty} \sum_{i=1}^{n} a_i\right)$

eg We know that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ and converges

$\sum_{n=1}^{\infty} \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (converges)

By comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

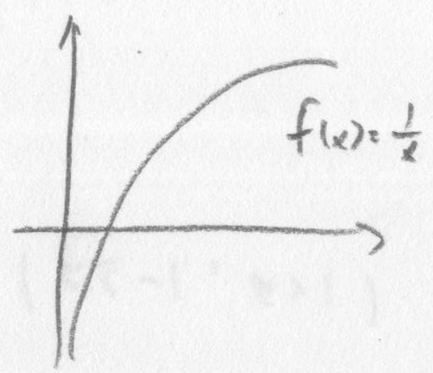
eg $\sum_{n=1}^{\infty} \frac{1}{n} \longrightarrow f(x) = \frac{1}{x}$

$x > 1$, $f(x) > 0$, decreasing

$\int_1^n \frac{1}{x}\,dx = \ln n - \ln 1 = \ln n$

$\lim_{n\to\infty} \ln n = \infty$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges



- Ratio Test

$a_n > 0$ $\quad \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

ρ < 1 $\sum a_n$ convergent

> 1 $\sum a_n$ diverges

$= 1$ Try other method

ex. $\sum \frac{2^n}{n!}$ $\quad a_n = \frac{2^n}{n!} > 0$

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{2}{n+1} \right| = 0 < 1$$

$\therefore \sum \frac{2^n}{n!}$ converges

Absolute Convergence (for non-positive series)

$\sum |a_n|$ converges $\qquad \sum |a_n| > \sum a_n$

$\Rightarrow \sum a_n$ converges

absolute convergent

ex $\sum a_n = 1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots$

$\sum |a_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots = \sum \frac{1}{n^2}$ (converges)

$\therefore \sum a_n$ converges also.

Alternating Series — Leibniz's test

$\{a_n\}$

① $a_n > 0$
② $a_1 \geq a_2 \geq a_3 \geq \cdots$ (numerically decreasing)
③ $a_n \to 0$ as $n \to \infty$

} we must fulfill all of these conditions if we want to use this method

then the alternating series is convergent.

ex $\frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \cdots$

$a_n = \frac{1}{\ln n} > 0$ and then decreasing numerically and $a_n \to 0$ as $n \to \infty$

$\therefore \sum (-1)^n \frac{1}{\ln n}$ converges.

$\sum_{n=1}^{\infty} a_n \longleftrightarrow \{a_n\}$

$\sum_{n=0}^{\infty} a_n x^n \longleftrightarrow \{a_n x^n\}$ Power Series

$\{a_n\}$ coefficient sequence

$\sum_{n=0}^{\infty} a_n (x - x_0)^n$

$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

$|x| < R$, $\sum a_n x^n$ converges

$|x| > R$, $\sum a_n x^n$ diverges

$|x| = R$, use the previous method to test which is convergent or not!

eg $\sum_{n=0}^{\infty} x^n \longrightarrow a_n = 1$

$R = \lim_{n\to\infty} \frac{1}{1} = 1$

- $|x| < 1$, $\sum x^n$ converges (eg. $\frac{1}{5^n}$ converges)
- $|x| > 1$, $\sum x^n$ diverges

$-1 \leq x < 1$ (convergent)

$x < -1$ or $x \geq 1$ (divergent)

For $x=1$, $\sum 1^n = \sum 1 = \infty$ (divergent)

For $x=-1$, $\sum (-1)^n = \sum 0 = 0$ (convergent)

$\therefore$ the interval of series is $[-1, 1)$

$1 - 1 + 1 - 1 + 1 - 1 + \cdots = 0$

$\sum a_n x^n = f(x)$ $\quad x \in (-R, R)$ — radius of convergence

$f(x)$ converges

1. $f'(x) = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n$

$\frac{d}{dx} \sum a_n x^n = \sum a_n \frac{dx^n}{dx}$ (convergent in $(-R, R)$)

2. $-R < \alpha \leq \beta < R$

$$\int_\alpha^\beta f(x)dx = \sum_{n=0}^{\infty} a_n \int_\alpha^\beta x^n dx = \sum_{n=0}^{\infty} a_n \frac{\beta^{n+1} - \alpha^{n+1}}{n+1}$$

$$\int_\alpha^\beta \sum a_n x^n dx = \sum \int_\alpha^\beta a_n x^n dx$$

Taylor Series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = f(x)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$